1. If A, B, C, and D are four nonempty sets, we define the product set or Cartesian product as 
\[ A \times B = \{(a, b) | a \in A \text{ and } b \in B\} \]
(a) Let \( A = \{1, 2, 3\} \) and \( B = \{r, s\} \), what is \( A \times B \)? (3 %)
(b) Prove that if C and D are countable, then \( C \times D \) is also countable. (12%)

2. How many distinct positive integers must you choose to guarantee that at least five of them will have the same remainder when divided by 7? Please prove your answer. (10%)

3. Please answer the following questions
(a) Draw a graph that has both an Euler circuit and a Hamiltonian circuit. (5%)
(b) Draw a graph that has an Euler circuit but has no Hamiltonian circuit. (5%)
(c) Draw a graph that has no Euler circuit but has a Hamiltonian circuit. (5%)
(d) Draw a graph that has neither an Euler circuit nor a Hamiltonian circuit. (5%)

4. Given a relation \( R \) on a set \( A \). For an integer \( n \geq 1 \), \( R^n \) is a new relation on \( A \) defined by \( aR^n b \) if and only if \( aRx_1, x_1Rx_2, \ldots, x_{n-1}Rx_b \) for the \( n+1 \) elements \( a, b, x_1, x_2, \ldots, x_{n-1} \) of \( A \). Please prove that \( R \) is transitive if and only if \( R^n \subseteq R \) for all \( n \geq 2 \). (20%)

5. We have 20 identical floppy disks to give away to 5 distinct students. There is no limit on how many disks each student may receive. Each student may receive as few as 0 and as many as 20 disks. It is also OK to have some or even all disks not given away. How many ways are there to make the distribution? (10%)
5. There is a row of N marbles. John is asked to remove the leftmost 1, 2, or 3 marbles at each time, until all marbles are removed.

(a) Let $A(N)$ be the number of ways John may perform this task. Please find a recurrence relation for $A(N)$ and compute $A(6)$. (5%)

(b) Let $P(N)$ be the number of ways in which John may perform this task in an odd number of steps; $Q(N)$ be the number of ways in which John may perform this task in an even number of steps. Find the recurrence relations for $P(N)$ and $Q(N)$ and compute $P(6)$ and $Q(6)$ to verify your answer in (a). (10%)

Let $A$ be the set $\{0, 1, 2, 3, 4, 5\}$. (10%)

(a) Please give an example of an equivalence relation $R$ defined on $A$ such that $A$ is divided into 3 equivalence classes, where not all equivalence classes have the same number of elements. You may describe the relation either as a set of ordered pairs, as a directed graph, or as a matrix.

(b) Find the transitive closure of the symmetric closure of the relation $R$ that you give in (a).